

# On the Paschos–Wolfenstein Relationship for Nuclei

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Nuclear effects and QCD perturbative corrections to the Paschos–Wolfenstein relationship are discussed. We argue that perturbative corrections largely cancel out in this relationship for total cross sections while the neutron excess correction in heavy nuclei is enhanced by Fermi motion and nuclear binding effects. These observations are discussed in the context of NuTeV measurement of the Weinberg mixing angle.

The scattering of (anti)neutrino from matter is mediated by charged  $W^+$  or  $W^-$  boson (charged current, CC), or by neutral  $Z$  boson (neutral current, NC). A relation between neutrino–antineutrino asymmetries in the NC and CC deep-inelastic (DIS) cross sections was derived long ago by Paschos and Wolfenstein [1]

$$R^- = \frac{\sigma_{\text{NC}}^\nu - \sigma_{\text{NC}}^{\bar{\nu}}}{\sigma_{\text{CC}}^\nu - \sigma_{\text{CC}}^{\bar{\nu}}} = \frac{1}{2} - \sin^2 \theta_W, \quad (1)$$

where  $\theta_W$  is the Weinberg mixing angle. The derivation of the Paschos–Wolfenstein relationship (PW) is solely based on the isospin symmetry and neglects contributions from heavy quarks. For this reason this relation is exact for an isoscalar target in a world without heavy quarks. In particular, this means that various strong interaction effects, including nuclear effects, should cancel out in  $R^-$  for an isoscalar target thus making Eq.(1) a very good tool for the measurement of the mixing angle in neutrino scattering.

However, in the real world the PW relation is subject to a number of corrections. In particular, it must be corrected for the effects due to possible  $s - \bar{s}$  and  $c - \bar{c}$  asymmetries in the target (see e.g. Refs. [2, 3]). Furthermore, the targets used in neutrino experiments are usually heavy nuclei, such as iron in NuTeV experiment [4]. Heavy nuclei typically have an excess of neutrons over protons and are non-isoscalar. For a non-isoscalar target Eq.(1) is not exact and receives various strong interaction corrections via admixture of the isovector components to  $R^-$ . In this paper we address QCD perturbative corrections and nuclear effects in the PW relationship for non-isoscalar nuclei.

We will discuss (anti)neutrino DIS in the leading twist (LT) QCD approximation. In this approximation the NC and CC structure functions are given in terms of parton distribution functions (PDFs). In order to simplify discussion of isospin effects, we consider the isoscalar,  $q_0(x) = u(x) + d(x)$ , and the isovector,  $q_1(x) = u(x) - d(x)$ , quark distributions (for simplicity, we suppress the explicit notation for the  $Q^2$  dependence of parton distributions). The calculation of the NC and CC cross sections, and the PW ratio in the leading

order in the strong coupling constant (LO) is straightforward. The next-to-leading order (NLO) correction to the PW relation is given in Ref.[2] and the next-to-next-to-leading order (NNLO) correction was calculated in Ref.[3]. The result can be written as<sup>a</sup>

$$\begin{aligned} R^- &= \frac{1}{2} - s_W^2 + \delta R^-, \\ \delta R^- &= \left[ 1 - \frac{7}{3}s_W^2 + \left( \frac{8\alpha_s}{9\pi} + 5.34 \frac{\alpha_s^2}{\pi^2} \right) \left( \frac{1}{2} - s_W^2 \right) \right] \left( \frac{x_1^-}{x_0^-} \right), \end{aligned} \quad (2)$$

where  $s_W^2 = \sin^2 \theta_W$ ,  $\alpha_s$  is the strong coupling, and  $x_a^- = \int dx x (q_a - \bar{q}_a)$ , with  $q_a$  and  $\bar{q}_a$  the distribution functions of quarks and antiquarks of type  $a$ . The subscripts 0 and 1 refer to the isoscalar  $q_0$  and isovector  $q_1$  quark distributions, respectively. In the derivation of Eq.(2) we expanded in  $x_1^-/x_0^-$  and retained only linear corrections. We also neglected contributions due to possible  $s - \bar{s}$  asymmetry discussion of which can be found in Refs.[2, 3, 6].

Equation (2) applies to any, not necessarily isoscalar, nuclear target. We observe that  $\delta R^-$  is determined by the valence part of the isovector quark distribution in the target. Complex nuclei, such as iron, have unequal number of neutrons ( $N$ ) and protons ( $Z$ ) and the isovector quark distribution is finite in such nuclei. In order to understand this effect, we first consider a simple approximation which is often used in processing of DIS data. In particular, we neglect nuclear effects and view the neutrino scattering off a nucleus as incoherent scattering off bound protons and neutrons at rest. We denote  $q_{a/T}$  as the distribution of quarks of type  $a$  in a target  $T$ . Then in considered approximation the nuclear distribution  $q_{a/A}$  is the sum of the corresponding quark distributions for bound protons and neutrons

$$q_{a/A} = Z q_{a/p} + N q_{a/n}. \quad (3)$$

We apply Eq.(3) to the isovector and isoscalar distributions assuming the isospin invariance of PDFs in the proton and neutron, i.e.  $q_{0/p}(x) = q_{0/n}(x)$  and  $q_{1/p}(x) = -q_{1/n}(x)$ . Then we have  $q_{0/A}(x) = A q_{0/p}(x)$  and  $q_{1/A}(x) = (Z - N) q_{1/p}(x)$ . For the ratio of average light cone momenta in the isovector and isoscalar states, which determine  $\delta R^-$  in Eq.(2), we obtain

$$(x_1/x_0)_A = -\delta N (x_1/x_0)_p, \quad (4)$$

where  $\delta N = (N - Z)/A$  is fractional excess of neutrons.

It follows from Eq.(4) that the neutron excess correction to  $R^-$  is negative for neutron-rich targets. In order to estimate the magnitude of this correction for iron target we first neglect  $\alpha_s$  terms in Eq.(2) and compute  $(x_1^-/x_0^-)_p = 0.45$  using the parton distributions of Ref.[7] at  $Q^2 = 20 \text{ GeV}^2$ . Keeping in mind application to NuTeV measurement [4] we use  $\delta N = 0.0574$  reported by NuTeV [5]. Then we have  $\delta R^- = -0.013$ . This is a large value on the scale of experimental errors of NuTeV measurement since  $|\delta R^-| \simeq 10\sigma$  (for

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<sup>a</sup>The total cross sections involve the integration of the structure functions over the full phase space of  $x$  and  $Q^2$ . Therefore  $\alpha_s$  and the moments  $x_i^-$  of the parton distributions are taken at some average scale  $Q^2$ , which has to be chosen according to specific experimental conditions. The NNLO coefficient in Eq.(2) is given in  $\overline{MS}$  scheme. We also remark that there is an error in the  $\alpha_s$  NLO coefficient in Refs.[2, 6]. I am grateful to K. McFarland for pointing this out.

a discussion of the neutron excess correction in this context see Ref. [6]). This gives us a motivation to study various corrections to  $R^-$  in more detail.

We first discuss perturbative corrections in Eq.(2) and consider the difference between NNLO and LO approximations

$$\Delta R^- = \delta R^-(\text{NNLO}) - \delta R^-(\text{LO}). \quad (5)$$

If we simply use the LO PDFs and  $\alpha_s$  of Ref. [7] at  $Q^2 = 20 \text{ GeV}^2$  we obtain from Eq.(2)  $\Delta R^- = -0.0008$ , which is about 6% of the LO value of  $\delta R^-$ . However this calculation is not fully consistent, since PDFs as well as the value of  $\alpha_s$  in Eq.(5) should correspond to the order of perturbative calculation. It is possible to take into account this effect using the results of analysis of Ref. [7], which provides PDFs to different order up to the NNLO approximation. If we do so we observe that the terms in the right side of Eq.(5) almost cancel each other leading to  $\Delta R^- \simeq 0.7 \cdot 10^{-4}$ . This value is the order of magnitude less than the result of a naive calculation (note also that the sign of the correction has changed). The reason for this cancellation is that  $\alpha_s$  terms in Eq.(2) turned out to be balanced by perturbative effects in PDFs which cause the decrease in the ratio  $x_1/x_0$  for valence quarks in the proton from 0.457 (LO) to 0.434 (NNLO).

In order to verify that this cancellation is not accidental we performed similar analysis for  $Q^2 = 10$  and  $100 \text{ GeV}^2$ . We have respectively  $\Delta R^- = -1.7 \cdot 10^{-5}$  and  $1.9 \cdot 10^{-4}$ . These values indicate that the cancellation seems to be systematic. It must be also noted that such a small value of  $\Delta R^-$  suggests that the magnitude of perturbative correction to  $R^-$  is within the variations of  $\delta R^-$  due to PDF uncertainties of Ref. [7]. Summarizing, we conclude that the LO calculation provides a good approximation of  $R^-$ .

Now we turn to the discussion of nuclear effects in  $R^-$ . In order to improve on Eq.(3), we consider nuclear binding and Fermi motion effects (for which we will use the abbreviation FMB) in terms of the convolution model of nuclear parton distributions (see, e.g., Refs. [8, 9, 10]). Then Eq.(3) should be replaced by

$$q_{a/A} = \langle q_{a/p} \rangle_p + \langle q_{a/n} \rangle_n, \quad (6)$$

where the two terms in the right side are the quark distributions in bound protons and neutrons averaged with the proton and neutron nuclear spectral functions, respectively. Similar equation can also be written for antiquark distribution. The explicit expression for the averaging in Eq.(6) is (see [9, 10])

$$x \langle q_{a/p} \rangle_p = \int d\varepsilon d^3k \mathcal{P}_p(\varepsilon, \mathbf{k}) \left(1 + \frac{k_z}{M}\right) x' q_{a/p}(x'), \quad (7)$$

$$x' = \frac{Q^2}{2k \cdot q} = \frac{x}{1 + (\varepsilon + k_z)/M}. \quad (8)$$

The integration in Eq.(7) is taken over the energy and momentum of bound protons (we separate the nucleon mass  $M$  from the nucleon energy  $k_0 = M + \varepsilon$ ). The quantity  $\mathcal{P}_p(\varepsilon, \mathbf{k})$  is the nuclear spectral function which describes the distribution of bound protons over the energy and momentum. In Eq.(7), the  $z$ -axis is chosen in the direction opposite to the momentum transfer  $q = (q_0, 0_\perp, -|\mathbf{q}|)$ , and  $x'$  is the Bjorken variable of the bound proton with four-momentum  $k$ . Equation similar to Eq.(7) also holds for neutrons with the obvious replacement of the spectral function and quark distributions. The spectral functions  $\mathcal{P}_p$  and  $\mathcal{P}_n$  are normalized to the proton and neutron number, respectively.

For the isoscalar and isovector nuclear parton distributions we obtain from Eq.(6)

$$q_{0/A} = \langle q_{0/p} \rangle_0, \quad (9a)$$

$$q_{1/A} = \langle q_{1/p} \rangle_1, \quad (9b)$$

where the averaging is respectively performed with isoscalar and isovector spectral functions,  $\mathcal{P}_0 = \mathcal{P}_p + \mathcal{P}_n$  and  $\mathcal{P}_1 = \mathcal{P}_p - \mathcal{P}_n$ .

The isoscalar and isovector spectral functions  $\mathcal{P}_0$  and  $\mathcal{P}_1$  are very different in complex nuclei. In an isoscalar nucleus with equal number of protons and neutrons one generally assumes vanishing  $\mathcal{P}_1$ <sup>b</sup> and nuclear effects are dominated by the isoscalar spectral function. In a nuclear mean-field model, in which a nucleus is viewed as Fermi gas of nucleons bound to self-consistent mean field, the spectral function can be calculated as

$$\mathcal{P}_{\text{MF}}(\varepsilon, \mathbf{p}) = \sum_{\lambda < \lambda_F} n_\lambda |\phi_\lambda(\mathbf{p})|^2 \delta(\varepsilon - \varepsilon_\lambda), \quad (10)$$

where  $\phi_\lambda(\mathbf{p})$  are the wave functions of the single-particle level  $\lambda$  in nuclear mean field and  $n_\lambda$  is the number of nucleons on this level. The sum in Eq.(10) runs over occupied single-particle levels with energies below the Fermi level  $\lambda_F$ . Equation (10) gives a good approximation to nuclear spectral function in the vicinity of the Fermi level, where the excitation energies of the residual nucleus are small. As separation energy  $|\varepsilon|$  becomes higher, Eq.(10) becomes less accurate. High-energy and high-momentum component of nuclear spectrum can not be described by the mean-field model and driven by correlation effects in nuclear ground state as witnessed by numerous studies. We denote this contributions to the spectral function as  $\mathcal{P}_{\text{cor}}(\varepsilon, \mathbf{p})$ .

In a generic nucleus the spectral function  $\mathcal{P}_1$  determines the isovector nucleon distribution. We now argue that the strength of  $\mathcal{P}_1$  is peaked about the Fermi surface. It is reasonable to assume that  $\mathcal{P}_{\text{cor}}$  is mainly isoscalar and neglect its contribution to  $\mathcal{P}_1$ . Then  $\mathcal{P}_1$  is determined by the difference of the proton and neutron mean-field spectral functions. If we further neglect small differences between the energy levels of protons and neutrons then  $\mathcal{P}_1$  will be determined by the difference in the level occupation numbers  $n_\lambda$  for protons and neutrons. Because of Pauli principle, an additional particle can join a Fermi system only on an unoccupied level. In a complex nucleus all but the Fermi level are usually occupied (the Fermi level has a large degeneracy factor). Therefore,  $\mathcal{P}_1$  is determined by the contribution from the Fermi level and we can write

$$\mathcal{P}_1 = (Z - N)|\phi_F(\mathbf{p})|^2 \delta(\varepsilon - \varepsilon_F), \quad (11)$$

where  $\varepsilon_F$  and  $\phi_F$  are the energy and the wave function of the Fermi level. In a nucleus with a large number of particles one can use the Fermi gas model to evaluate the wave function  $\phi_F$ . In this model  $|\psi_F(p)|^2 \propto \delta(p_F - p)$ , where  $p_F$  is the Fermi momentum, and we have

$$\mathcal{P}_1 = (Z - N)\delta(p - p_F)\delta(\varepsilon - \varepsilon_F)/(4\pi p_F^2). \quad (12)$$

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<sup>b</sup>It must be commented that this statement is violated by a number of effects even in the  $Z = N$  nuclei. The finite difference between the proton and neutron spectral functions is generated by Coulomb interaction and isospin-dependent effects in the nucleon–nucleon interaction. The discussion of these effects goes beyond the scope of this paper and we leave this topic for future studies.

We now apply these equations to calculate the binding and momentum distribution effects on average quark light-cone momenta in the isoscalar and isovector quark distributions. Integrating Eq.(7) over  $x$  and keeping the terms to order  $\varepsilon/M$  and  $\mathbf{k}^2/M^2$  we have

$$\frac{x_{0/A}}{A} = \left(1 + \frac{\varepsilon_0 + \frac{2}{3}T_0}{M}\right)x_{0/p}, \quad (13a)$$

$$\frac{x_{1/A}}{A} = -\delta N \left(1 + \frac{\varepsilon_F + \frac{2}{3}T_F}{M}\right)x_{1/p}, \quad (13b)$$

where  $\varepsilon_0$  and  $T_0$  are the separation and kinetic energy per nucleon averaged with the isoscalar nuclear spectral function  $\mathcal{P}_0$  and  $T_F = p_F^2/(2M)$ . In order to quantitatively estimate this effect we observe that the energy of the Fermi level  $\varepsilon_F$  equals the minimum nucleon separation energy. For the iron nucleus we take  $\varepsilon_F = -10\text{MeV}$  and  $p_F = 260\text{MeV}$  (the corresponding energy  $T_F = 36\text{ MeV}$ ). In order to calculate the isoscalar parameters  $\varepsilon_0$  and  $T_0$  we use the model spectral function of Ref.[11] which takes into account both the mean-field and correlated contributions (see also [12]). We find that the naive neutron excess correction by Eq.(4) should be increased by the factor 1.055.

We now discuss these results in the context of NuTeV effect [4]. We assume that the weak mixing angle can be calculated from Eq.(2) in terms of experimental  $R^-$ . In particular, we are interested in the variation of  $s_W^2$  because of nuclear effects and effects of higher order in  $\alpha_s$ , since NuTeV analysis was carried out in LO approximation without nuclear effects. The correction  $\Delta s_W^2$  is apparently given by the difference between the corrected and uncorrected expressions for  $\delta R^-$

$$\Delta s_W^2 = \delta R^-(\text{NNLO+FMB}) - \delta R^-(\text{LO}). \quad (14)$$

Perturbative corrections are largely canceled out in the difference as discussed above and the resulting value  $\Delta s_W^2 = -0.00065$  is practically saturated by nuclear effects. With certain care this value can be viewed as a correction to the value of the Weinberg angle  $\sin^2 \theta_W$  measured by NuTeV [4].<sup>c</sup>

In summary, we discussed perturbative QCD corrections to the PW relationship together with nuclear binding and Fermi motion effects. A cancellation of QCD perturbative corrections to the PW relationship for the total cross sections has been observed. We found a negative correction due to nuclear effects to the PW relationship for the total cross sections for neutron-rich targets and estimated this effect on the Weinberg angle of NuTeV measurement.

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<sup>c</sup>It should be remarked in this context that total cross sections were not measured in NuTeV experiment. For this reason a more detailed analysis of differential cross sections with the proper treatment of experimental cuts and (anti)neutrino flux is needed.

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